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A rational approach to Hopf rings

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Abstract

We study Hopf rings of the form $E_*(\boldsymbol{F}_*)$, where $E_*(-)$ is a complex oriented homology theory and $\boldsymbol{F}_*$ is the graded ring space arising from a complex oriented Ω-spectrum F. We propose a new model Hopf ring $E^{Q}_*(\boldsymbol{F}_*)$ which is constructed by purely algebraic means from the rationalisations $E\mathbb{Q}_*(\boldsymbol{F}_*)$, $E_*(\boldsymbol{F}\mathbb{Q}_*)$ and $E\mathbb{Q}_*(\boldsymbol{F}\mathbb{Q}_*)$, and which in many cases is isomorphic to $E_*(\boldsymbol{F}_*)$. Our model may be expressed in terms of the conjugate Bell polynomials associated with the exponential series for E and F, and we explain how calculations may be carried out in this context. We discuss the relationship between $E^{Q}_*(\boldsymbol{F}_*)$ and the well-known model $E^{R}_*(\boldsymbol{F}_*)$ due to Ravenel and Wilson.

1. Introduction

Let F be a multiplicative, complex oriented spectrum, whose coefficient ring $F_* = F^{-*} = \pi_*(F)$ is torsion free and concentrated in even dimensions. As detailed in [1], the free F^* module $F^*(\mathbb{C}P^\infty)$ consists of formal power series in the orientation class x^F, whose powers define dual basis elements β_n in $F_{2n}(\mathbb{C}P^\infty)$. Write $\{\boldsymbol{F}_r : r \in \mathbb{Z}\}$ for the spaces in the Ω-spectrum associated to F, so that $\boldsymbol{F}_r$ represents the cohomology group $F^r(-)$, and $\Omega\boldsymbol{F}_r$ is homotopy equivalent to $\boldsymbol{F}_{r-1}$. In particular, x^F is represented by a map

$$x^F : \mathbb{C}P^\infty \to \boldsymbol{F}_2,$$

whose pullback under the product map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ defines the formal group law $F(X, Y)$ for F.

Suppose given a second such spectrum, E. Provided that the spaces $\boldsymbol{F}_r$ enjoy a Künneth isomorphism

$$E_*(\boldsymbol{F}_r) \otimes_{E_*} E_*(\boldsymbol{F}_r) \cong E_*(\boldsymbol{F}_r \times \boldsymbol{F}_r),$$

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then the E_* homology module $E_*(\boldsymbol{F}_*)$ constitutes a ring object in the category of graded E_* coalgebras, and hence forms a *Hopf ring*. This concept was first formalised by Ravenel and Wilson, who computed the structure of $E_*(\boldsymbol{F}_*)$ in several important cases [6]; we follow their notation as closely as possible below. In particular, we take all homology and cohomology theories to be unreduced unless otherwise indicated, and we write $\boldsymbol{F}_r'$ for the basepoint component of $\boldsymbol{F}_r$.

Any coefficient element f in F^{-r} is represented by a map of the one-point space into $\boldsymbol{F}_{-r}$, and we let $[f]$ in $E_0(\boldsymbol{F}_{-r})$ denote the image of 1 under this map; it is a grouplike element, its image under the coproduct map being $[f] \otimes [f]$. The Hopf ring generated over E_* by all such $[f]$ is then denoted by $E_*[F^*]$, reflecting the fact that it is the ring ring of F^* over E_*.

Write e in $E_1(\boldsymbol{F}_1)$ for the homology suspension of $[1]$ in $E_0(\boldsymbol{F}_0)$, and b_n for the image $x_*^F(\beta_n)$ in $E_{2n}(\boldsymbol{F}_2)$, noting that $b_0 = [0]$ and $b_1 = e \circ e$. Clearly e is primitive, and the coproduct formula

$$\delta(b_n) = \sum b_{n-i} \otimes b_i \tag{1.1}$$

is inherited from $\mathbb{C}P^\infty$. Ravenel and Wilson form the free Hopf ring over $E_*[F^*]$ generated by e and the b_n, and obtain their model $E_*^R(\boldsymbol{F}_*)$ by identifying b_0 with $[0]$ and b_1 with $e \circ e$, and then introducing the relations which arise from the interaction between the formal group laws for E and F, respectively.

These relations may be expressed in terms of generating functions in variables s and t by

$$b(s +_E t) = b(s) +_{[F]} b(t), \tag{1.2}$$

where $b(s) = \sum_{i \geq 0} b_i s^i$ and where $+_E$ and $+_{[F]}$ denote formal sums corresponding to the respective formal group laws (see [6, 3.8(i)]). However, the elegance of such formulae disguises their enormous underlying complexity. For example, in any given case it may be far from simple to determine whether or not $E_*^R(\boldsymbol{F}_*)$ contains torsion. Nevertheless, it is straightforward to define a natural Hopf ring map

$$\tau_F : E_*^R(\boldsymbol{F}_*) \longrightarrow E_*(\boldsymbol{F}_*)$$

and Ravenel and Wilson prove that if F is MU or the Brown Peterson spectrum BP then τ_F is an isomorphism. In these cases, therefore, the model is precise, although the obstacles to effective computation remain.

They may sometimes be avoided in practice by resorting to a simpler set of relations based on the p series $[p]_F(X)$, induced on x^F by the pth power self-map of $\mathbb{C}P^\infty$ for a suitable prime p. The alternative relations require a single variable s, and may be expressed as

$$b([p]_E(s)) = [p]_F(b(s)) \tag{1.3}$$

(see [6, 3.8(ii)]). In favourable cases, such as when F is p-typical, (1.3) determines $E_*^R(\boldsymbol{F}_*)$ almost entirely, and many useful calculations may be carried out with its help.

The motivating theme of our work here is to find an alternative simplification of (1.2), based on rationalisation rather than p-typification. Since rationalisation is used in most of the existing Hopf ring computations (including those in [6]), there is plenty of evidence to suggest that such an alternative should be widely applicable.

We consider the exponential series

$$\exp^F(x) = \sum_{i \geq 1} \frac{1}{i!} f_{i-1} x^i$$

in the rational cohomology ring $F\mathbb{Q}^*(\mathbb{C}P^\infty)$, and construct a new (and torsion free) model Hopf ring $E^Q_*(F_*)$ in terms of certain corresponding rational relations. We offer evidence that calculations in $E^Q_*(F_*)$ may be effected with moderate ease in terms of conjugate Bell polynomials (see [9]), in the rational environment provided by $E\mathbb{Q}_*(F\mathbb{Q}_*)$. Moreover, we prove the existence of a natural commutative diagram

$$\begin{array}{ccc} E^R_*(F_*) & \xrightarrow{\tau_F} & E_*(F_*) \\ \sigma_F \downarrow & & \downarrow \pi_F \\ E^Q_*(F_*) & \xrightarrow{\kappa_F} & E_*(F_*)/T, \end{array}$$

where π_F is projection onto the torsion free quotient Hopf ring, and κ_F is a suitably defined Hopf ring monomorphism.

By modifying and extending the work of Chan [2], we demonstrate that both π_F and κ_F are isomorphisms for all the major cases considered by Ravenel and Wilson. Crucially, however, we use neither the methods nor the results of [6], and so achieve considerable simplification in the theory. This validates the use of $E^Q_*(F_*)$ as a model, and circumvents discussion of some of the thornier aspects of the relations (1.2). A typical example is provided by recent joint work of the first author concerning the cases when F is Landweber exact (such as the elliptic cohomology spectrum); the effort devoted in [5] to elucidating the structure of $E^R_*(F_*)$, and the attendant reliance on the results of [6], may all be avoided if the model $E^Q_*(F_*)$ is used instead.

We begin in Section 2 by decomposing the graded ring space $F\mathbb{Q}_*$ into Eilenberg–MacLane spaces, and introducing our rational relations in $E_*(F\mathbb{Q}_*)$. In Section 3 we consider a commutative square linking the rational Hopf rings $E_*(F\mathbb{Q}_*)$, $E\mathbb{Q}_*(F_*)$, and $E\mathbb{Q}_*(F\mathbb{Q}_*)$, and determine their structure with the help of Section 2. As a result, we may define our model $E^Q_*(F_*)$ as a sub-Hopf ring of $E\mathbb{Q}_*(F\mathbb{Q}_*)$ in Section 4, where we derive some of its basic properties and explain some fundamental examples, considering in particular the cases where F is Landweber exact.

In Section 5 we turn to more detailed computations with $E^Q_*(F_*)$ in terms of Bell polynomials, and by way of illustration we consider the case when F is the spectrum K of complex K-theory; although straightforward, this is not covered in [6]. We devote our concluding Section 6 to explaining connections between $E^Q_*(F_*)$ and the Ravenel and Wilson model $E^R_*(F_*)$.

It is important to clarify our assumptions on the coefficient ring F^*, which we have already required to be torsion free and concentrated in even dimensions. These conditions are sufficient to validate the results of Section 2, but to avoid having to strengthen them in later sections we shall henceforth follow [5] and also insist that F^* is a free R-module of countable rank over some subring R of $\mathbb{Q}$.

We must comment on four items of notation which we often use below.

Firstly, we write w for the generator of $\pi_2(\mathbb{C}P^\infty)$ induced by inclusion of the bottom cell, and for its Hurewicz image in any homology group $G_2(\mathbb{C}P^\infty)$ (including stable homotopy); thus the powers w^n lie in $G_{2n}(\mathbb{C}P^\infty)$ for each $n \geq 1$. In the case when G is a complex oriented spectrum E, then w coincides with the element β_1 introduced above, and we shall also write w for its image b_1 in $E_2(F_2)$ given any complex oriented F. This duplication of labels is potentially ambiguous; but we believe that the context invariably identifies the class under consideration, and that the notation required to differentiate fully between the cases would be unpleasantly complicated. Our choice of the symbol w is governed by the fact that the element it represents appears in several polynomial expressions, and it is a convenient mnemonic to think of it as a variable.

Secondly, we write e for the homology suspension of the element [1] in several different situations. They are, however, all closely related (for example by rationalisation), and correspond to each other sufficiently well under the obvious natural maps that attempting to differentiate between them would also be an unnecessary en-ncumbrance. Similar remarks apply to the elements b_n.

Thirdly, we often write $H(A,r)$ for the Eilenberg–MacLane space representing the cohomology group $H^r(-,A)$, where the abelian group A may consist of the elements of fixed grading in some larger graded ring. This conflicts with our standard notation $\boldsymbol{HA}_r$, but is usually more manageable, and has the additional virtue of being familiar to traditionalists.

Fourthly, we follow the growing trend of using juxtaposition to denote loop sums in all Hopf rings; as a result, we incorporate a product sign, rather than the usual $*$, into several of our more intricate formulae.

2. The rational relations

In this section we study the graded ring space $\boldsymbol{F\mathbb{Q}}_*$ and introduce our rational relations in $E_*(\boldsymbol{F\mathbb{Q}}_*)$; they are based on the exponential series for E. We begin by reviewing some classical ideas on the exponential map, the Boardman homomorphism, and rational Eilenberg–MacLane spaces.

It is important to note that our assumptions on the coefficients F^* imply that the cohomology ring $F^*(Z)$ embeds in its rationalisation $F\mathbb{Q}^*(Z)$ for any space or spectrum Z with cells only in even (or only in odd) dimensions. In this situation, we often use the same label for an integral cohomology element and its rationalisation, particularly with reference to the complex orientation class x^F.

By the same token, there is a strict isomorphism between the formal group laws $X + Y$ and $F(X, Y)$ over $F\mathbb{Q}$, effected by means of a formal power series

$$\exp^F(X) = \sum_{i \geq 1} \frac{1}{i!} f_{i-1} X^i$$

for some f_{i-1} in F^{2-2i}. By definition, this is the *exponential series* of F, and we wish to interpret it geometrically using a convenient variant of a method pioneered by Adams [1].

Given any spectrum F, the Boardman homomorphism (as expounded in [7], for example) may be applied to the rationalisation $F\mathbb{Q}$ to yield an isomorphism

$$F\mathbb{Q}^*(-) \cong H^*(-; F\mathbb{Q}^*) \cong H^*(-) \hat{\otimes} F\mathbb{Q}^* \tag{2.1}$$

of graded cohomology rings. Restricting to $F^*(-)$ yields the Chern–Dold character [4], best known in the case when F is the complex K-theory spectrum K.

Applying (2.1) to $\mathbb{C}P^\infty$, we obtain

$$F\mathbb{Q}^*(\mathbb{C}P^\infty) \cong F\mathbb{Q}^*[\![x]\!], \tag{2.2}$$

where x denotes the complex orientation x^H of integral cohomology induced by the identity map of $\mathbb{C}P^\infty$. The isomorphism (2.2) expresses x^F in $F\mathbb{Q}^2(\mathbb{C}P^\infty)$ as a certain formal power series in x, which the Boardman homomorphism identifies as

$$\sum_{i \geq 1} \frac{1}{i!} f_{i-1} x^i \in F\mathbb{Q}^*[\![x]\!], \tag{2.3}$$

where f_{i-1} is the image of w^i under the homomorphism induced in stable homotopy by $x^F : \mathbb{C}P^\infty \to S^2F$. Henceforth we take (2.3) to be our *definition* of the exponential series $\exp^F(x)$.

There is a decomposition

$$F\mathbb{Q}_r \cong \prod_{j \geq -r} H(F\mathbb{Q}_j, j + r) \tag{2.4}$$

of the rational H-space $F\mathbb{Q}_r$ into Eilenberg–MacLane spaces; this represents the multiplicative isomorphism (2.1), and so is an equivalence of graded ring spaces. It may be further refined by choosing vector space generators for each $F\mathbb{Q}_j$ over $\mathbb{Q}$, although care must be taken if F_r is not of finite type. Amongst such cases, to which we shall refer in Section 4, are elliptic cohomology and the Johnson–Wilson spectra $E(n)$, for $n > 1$.

For the moment, we restrict our attention to the identity element 1 in $F\mathbb{Q}_0$. Often this generates all of $F\mathbb{Q}_0$, and in general it splits off a copy of $\mathbb{Q}$, thereby defining a factor $H(\mathbb{Q}, s)$ of each term $H(F\mathbb{Q}_0, s)$ in (2.4). We compose the inclusion of such a factor with a representative for x^r in $H^{2r}(\mathbb{C}P^\infty)$, so obtaining a map

$$\mathbb{C}P^\infty \to H(\mathbb{Q}, 2r) \subset F\mathbb{Q}_{2r}. \tag{2.5}$$

We emphasise the important fact that $H(\mathbb{Q}, 2r) \subset \boldsymbol{F}\mathbb{Q}_{2r}$ is an H-map with respect to loop sum. Further reference to the Boardman homomorphism confirms that (2.5) represents x^r in $F\mathbb{Q}^{2r}(\mathbb{C}P^\infty)$ under (2.1), and we therefore also label it x^r.

Having now described both x and x^F as maps $\mathbb{C}P^\infty \to \boldsymbol{F}\mathbb{Q}_2$, we may use them in conjunction with the exponential series (2.3) to obtain our promised relations.

Consider the generating function $\beta(s) = \sum_{i \geq 0} \beta_i s^i$, and following Section 1 write $b(s) = x_*^F(\beta(s)) = \sum_{i \geq 0} b_i s^i$. Applying $E_*(-)$ to (2.3) yields

$$b(s) = \prod_{i \geq 1} \left\{ \left[\frac{f_{i-1}}{i!} \right] \circ (x_*(\beta(s)))^{\circ i} \right\}, \tag{2.6}$$

recalling our convention that $\prod$ denotes iterated loop sum. These relations will provide vital insight into the elements b_n of $E_*(\boldsymbol{F}\mathbb{Q}_*)$, and enable us to perform computations in our model Hopf ring.

3 Rational structure

In this section, we employ the results of Section 2 to analyse the commutative square of Hopf ring homomorphisms

$$\begin{array}{ccc} E_*(\boldsymbol{F}_*) & \longrightarrow & E_*(\boldsymbol{F}\mathbb{Q}_*) \\ \downarrow & & \downarrow \\ E\mathbb{Q}_*(\boldsymbol{F}_*) & \longrightarrow & E\mathbb{Q}_*(\boldsymbol{F}\mathbb{Q}_*), \end{array} \tag{3.1}$$

where the horizontal maps are induced by the rationalisation of F, and the vertical maps by the rationalisation of E. This diagram is essential to the definition of our model.

Our initial aim is to investigate $E_*(\boldsymbol{F}\mathbb{Q}_*)$, and we begin with the case $H_*(\boldsymbol{H}\mathbb{Q}_*)$ whose determination in Proposition 3.1 below is surely known to experts. Nevertheless we sketch the proof, since it helps to set the scene for subsequent calculations. Recall that e in $E_1(\boldsymbol{F}_1)$ denotes the homology suspension of $[1]$ in $E_0(\boldsymbol{F}_0)$, that $a \circ e$ is the homology suspension of an arbitrary element a, and that w in $E_2(\boldsymbol{F}_2)$ denotes $e^{\circ 2}$.

Proposition 3.1. *The Hopf ring $H_*(\boldsymbol{H}\mathbb{Q}_*)$ is free over $\mathbb{Z}[\mathbb{Q}]$ on the primitive generator e.*

Proof. We first describe the structure of the stated free Hopf ring, noting that the element e and all its odd $\circ$ powers are exterior, by virtue of being odd dimensional. The zero graded part $\mathbb{Z}[\mathbb{Q}]$ is concentrated entirely in dimension zero, and the Hopf ring distributivity axiom implies that its action on e satisfies

$$[p+q] \circ e = [p] \circ e + [q] \circ e \tag{3.2}$$

for any rationals p and q. The same axiom also shows that

$$(w^{\circ r})^m \circ (w^{\circ s})^n = \begin{cases} m!(w^{\circ(r+s)})^m & \text{if } m = n, \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

for all positive integers r, s, m and n. Thus the part of grading $r > 0$ is the free commutative graded algebra on the single generator $e^{\circ r}$ over $\mathbb{Q}$; in other words, it is polynomial when r is even, and exterior when r is odd.

We next turn to the Hopf ring $H_*(H\mathbb{Q}_*)$, observing that $H(\mathbb{Q}, 0)$ is the discrete space $\mathbb{Q}$, with $*$ and $\circ$ products given by the standard addition and multiplication. Thus $H_*(H(\mathbb{Q}, 0))$ is indeed the ring ring $\mathbb{Z}[\mathbb{Q}]$, and acts on e as required by (3.2). In similar vein, $H(\mathbb{Q}, 1)$ is a rational circle; thus it has exterior homology generated by the single element e over $\mathbb{Q}$.

Now consider $H(\mathbb{Q}, r)$ as the classifying space $BH(\mathbb{Q}, r - 1)$, for all $r > 1$. We may then repeatedly apply the Rothenberg–Steenrod spectral sequence

$$E^2 = \mathrm{Tor}^{H_*(H(\mathbb{Q}, r-1))}(\mathbb{Z}, \mathbb{Z}) \Rightarrow H_*(H(\mathbb{Q}, r)),$$

bearing in mind that whenever W is a free graded commutative algebra over $\mathbb{Q}$, then so is $\mathrm{Tor}^W(\mathbb{Z}, \mathbb{Z})$, generated by elements corresponding to the homology suspensions of the indecomposables $Q(W)$. Each spectral sequence collapses for dimensional reasons, and $H_*(H\mathbb{Q}_*)$ therefore has the required structure. We note that the relations (3.3) may readily be obtained geometrically by a standard computation with the cup product map

$$H(\mathbb{Q}, 2r) \wedge H(\mathbb{Q}, 2s) \to H(\mathbb{Q}, 2(r + s)). \qquad \square$$

We extend this result by recalling the decomposition (2.5).

Proposition 3.2. *The Hopf ring $H_*(F\mathbb{Q}_*)$ is free over $\mathbb{Z}[F\mathbb{Q}^*]$ on the primitive generator e.*

Proof. Let us first consider $H_*(F\mathbb{Q}_r)$, the part of grading r, for any r. The zero dimensional elements all lie in the factor $H_*(H(F\mathbb{Q}_{-r}, 0))$, and constitute the group ring $\mathbb{Z}[F\mathbb{Q}^{-r}]$. Now the distributivity axiom implies that

$$[f] \circ (e^{\circ(j+r)})^m = ([f] \circ e^{\circ(j+r)})^m$$

for every f in $F\mathbb{Q}^j$ and every positive integer m; so for $j > -r$ each factor $H_*(H(F\mathbb{Q}_j, j + r))$ contributes the free graded commutative algebra on $F\mathbb{Q}_j$, whose elements have been suitably suspended by applying $\circ e^{\circ(j+r)}$.

Assembling these observations for each value of r, we deduce that the zero dimensional elements of $H_*(F\mathbb{Q}_*)$ constitute the ring ring $\mathbb{Z}[F\mathbb{Q}^*]$, whilst the positive dimensional elements contribute copies of the free graded commutative algebra on

$F\mathbb{Q}_*$, one generated by each e^{os} for $s > 0$. Moreover, the action of the ring ring on e satisfies

$$[f+g]\circ e = [f]\circ e + [g]\circ e \tag{3.4}$$

for each f and g in $F\mathbb{Q}^r$, as required. $\square$

Our first structure theorem now follows easily.

Theorem 3.3. *The Hopf ring $E_*(F\mathbb{Q}^*)$ is free over $E_*[F\mathbb{Q}^*]$ on the primitive generator e.*

Proof. We have that

$$E_*(F\mathbb{Q}_*) \cong E_* \otimes H_*(F\mathbb{Q}_*),$$

because the relevant Atiyah–Hirzebruch spectral sequence collapses. Also, the free Hopf ring over $E_*[F\mathbb{Q}^*]$ is obtained from the free Hopf ring over $\mathbb{Z}[F\mathbb{Q}^*]$ by applying $E_* \otimes$; so the proof follows directly from Proposition 3.2. $\square$

Corollary 3.4. *The evenly graded Hopf ring $E_*(F\mathbb{Q}_{2*})$ is free over $E_*[F\mathbb{Q}^*]$ on the primitive generator w.*

Note that Theorem 3.3 and Corollary 3.4 apply in particular to $E\mathbb{Q}_*(F\mathbb{Q}_*)$, the lower right-hand corner of diagram (3.1). We therefore further deduce the following corollary.

Corollary 3.5. *The Hopf ring $E_*(F\mathbb{Q}_*)$ is a sub-Hopf ring of $E\mathbb{Q}_*(F\mathbb{Q}_*)$.*

Proof. It suffices to remark that $E_*[F\mathbb{Q}^*]$ is a sub-Hopf ring of $E\mathbb{Q}_*[F\mathbb{Q}^*]$. $\square$

Continuing our study of diagram (3.1), we turn our attention to $E\mathbb{Q}_*(F_*)$, once more beginning with a special case.

Proposition 3.6. *The Hopf ring $H\mathbb{Q}_*(F_*)$ is free over $\mathbb{Q}[F^*]$ on the primitive generator e.*

Proof. Let the map $\rho: F_* \to F\mathbb{Q}_*$ represent rationalisation $F^*(-) \to F\mathbb{Q}^*(-)$. When restricted to connected components, it yields a map $\rho': F'_* \to F\mathbb{Q}'_*$ of graded Hopf spaces, which by definition rationalises homotopy groups. Therefore by standard localisation arguments (such as in [11], for example), it induces an isomorphism in $H\mathbb{Q}_*(-)$.

We deduce that the Hopf ring homomorphism

$$\rho_*: H\mathbb{Q}_*(F_*) \to H\mathbb{Q}_*(F\mathbb{Q}_*)$$

is an isomorphism in nonzero dimensions, and is the inclusion $\mathbb{Q}[F^*] \subset \mathbb{Q}[F\mathbb{Q}^*]$ of ring rings in dimension zero; it therefore remains for us to prove that our stated free Hopf ring agrees with $H\mathbb{Q}_*(F\mathbb{Q}_*)$ in positive dimensions.

However, from Theorem 3.3 $H\mathbb{Q}_*(F\mathbb{Q}_*)$ is the free Hopf ring over $\mathbb{Q}[F\mathbb{Q}^*]$ generated by e, and

$$[f] \circ e = \left[n\frac{f}{n}\right] \circ e = n\left[\frac{f}{n}\right] \circ e$$

for all f in F^*, and any nonzero integer n. Hence

$$\left[\frac{f}{n}\right] \circ e = \frac{1}{n}[f] \circ e \tag{3.5}$$

and our result is established. □

Just as in Theorem 3.3, we may extend Proposition 3.6 because the relevant Atiyah–Hirzebruch spectral sequence collapses.

Theorem 3.7. *The Hopf ring $E\mathbb{Q}_*(F_*)$ is free over $E\mathbb{Q}_*[F^*]$ on the primitive generator e.*

Mimicking Corollary 3.4, we deduce the following corollary.

Corollary 3.8. *The evenly graded Hopf ring $E\mathbb{Q}_*(F_{2*})$ is free over $E\mathbb{Q}_*[F^*]$ on the primitive generator w.*

Note that both Theorem 3.7 and Corollary 3.8 also apply to $E\mathbb{Q}_*(F\mathbb{Q}_*)$, and give the expected results consistent with Theorem 3.3 and Corollary 3.4. In tandem with Corollary 3.5 we have the following Corollary.

Corollary 3.9. *The Hopf ring $E\mathbb{Q}_*(F_*)$ is a sub-Hopf ring of $E\mathbb{Q}_*(F\mathbb{Q}_*)$.*

Proof. It suffices to remark $E\mathbb{Q}_*[F^*]$ is a sub-Hopf ring of $E\mathbb{Q}_*[F\mathbb{Q}^*]$. □

At this point it is worth pausing to reflect on the distinction between $E_*(F\mathbb{Q}_*)$, $E\mathbb{Q}_*(F_*)$, and $E\mathbb{Q}_*(F\mathbb{Q}_*)$. The first two are sub-Hopf rings of the third, and all are freely generated by e, over $E_*[F\mathbb{Q}^*]$, $E\mathbb{Q}_*[F^*]$, and $E\mathbb{Q}_*[F\mathbb{Q}^*]$, respectively. The distinction is therefore encapsulated by the actions of these ring rings on e, which are documented in (3.4) and (3.5) above. We deduce that

$$E_*[F\mathbb{Q}^*] \circ e = E\mathbb{Q}_*[F^*] \circ e = E\mathbb{Q}_*[F\mathbb{Q}^*] \circ e$$

in $E\mathbb{Q}_*(F\mathbb{Q}_*)$, and that all three are isomorphic to $E_* \otimes F_* \otimes \mathbb{Q}$. Since the homology suspension homomorphism $\circ e$ annihilates every $*$-decomposable element, these formulae may be interpreted in the light of Theorems 3.3 and 3.7 as a calculation of the indecomposable quotients of the three ring rings.

In short, the difference between $E\mathbb{Q}_*(\boldsymbol{F}_*)$, $E_*(\boldsymbol{F}\mathbb{Q}_*)$, and $E\mathbb{Q}_*(\boldsymbol{F}\mathbb{Q}_*)$ lies entirely in the respective ring rings, and vanishes as soon as we suspend once. This is the precursor of the stable situation, where

$$E\mathbb{Q}_*(F) = E_*(F\mathbb{Q}) = E\mathbb{Q}_*(F\mathbb{Q})$$

and all are isomorphic to $E_* \otimes F_* \otimes \mathbb{Q}$; it is of some interest that stability is achieved after a single suspension. In fact many of our methods may also be applied to illuminate the stable case, a theme which we are currently developing elsewhere [3].

We may therefore completely describe diagram (3.1) in terms of the commutative square of ring rings

$$\begin{tikzcd} E_*[F^*] \arrow[r] \arrow[d] & E_*[F\mathbb{Q}^*] \arrow[d] \\ E\mathbb{Q}_*[F^*] \arrow[r] & E\mathbb{Q}_*[F\mathbb{Q}^*] \end{tikzcd} \tag{3.6}$$

and the following description of the remainder.

Proposition 3.10. *In* $E\mathbb{Q}_*(\boldsymbol{F}\mathbb{Q}_*)$, *the three subalgebras* $\tilde{E}_*(\boldsymbol{F}\mathbb{Q}'_*)$, $\widetilde{E\mathbb{Q}}_*(\boldsymbol{F}'_*)$ *and* $\widetilde{E\mathbb{Q}}_*(\boldsymbol{F}\mathbb{Q}'_*)$ *are all equal.*

In general this result fails if we substitute unreduced homology or nonconnected versions of the ring spaces.

To highlight the distinction between the ring rings in (3.6), let us consider the simplest case, namely when E and F are both the integral Eilenberg–MacLane spectrum H. Then $E_*[F\mathbb{Q}^*]$ becomes $\mathbb{Z}[\mathbb{Q}]$, whilst $E\mathbb{Q}_*[F^*]$ becomes $\mathbb{Q}[\mathbb{Z}]$ and $E\mathbb{Q}_*[F\mathbb{Q}^*]$ becomes $\mathbb{Q}[\mathbb{Q}]$. No two of these are isomorphic, even as group rings.

4. The model and examples

We devote this section to introducing our model Hopf ring in terms of the constructions of Section 3. We recall that b_n denotes the element $x_*^F(\beta_n)$ in $E_{2n}(\boldsymbol{F}_2)$, which therefore passes to the corresponding b_n in each of the Hopf rings of diagram (3.1).

Definition 4.1. The *model Hopf ring* $E_*^{\mathbb{Q}}(\boldsymbol{F}_*)$ is defined to be the sub-Hopf ring of $E\mathbb{Q}_*(\boldsymbol{F}\mathbb{Q}_*)$ generated by $E_*[F^*]$, the element e, and the elements b_n for all $n \geq 0$.

Note that the considerations of Section 3 automatically imply that $E_*^{\mathbb{Q}}(\boldsymbol{F}_*)$ is a sub-Hopf ring of both $E_*(\boldsymbol{F}\mathbb{Q}_*)$ and $E\mathbb{Q}_*(\boldsymbol{F}_*)$, and as such may be described in exactly the same terms.

We may therefore deduce that the model is precise in certain basic rational situations.

Proposition 4.2. *The Hopf rings $E_*(F\mathbb{Q}_*)$ and $E\mathbb{Q}_*(F_*)$ coincide with their respective models $E_*^{\mathbb{Q}}(F\mathbb{Q}_*)$ and $E\mathbb{Q}_*^{\mathbb{Q}}(F_*)$.*

Proof. Since $F\mathbb{Q}\mathbb{Q}_* = F\mathbb{Q}_*$, the first model is contained in $E_*(F\mathbb{Q}_*)$ by definition; moreover, e is amongst its generator over $E_*[F\mathbb{Q}^*]$, so by Theorem 3.3 it consists of the entire Hopf ring.

Since $E\mathbb{Q}\mathbb{Q}_*(-) = E\mathbb{Q}_*(-)$, the second model is contained in $E\mathbb{Q}_*(F_*)$ by definition; moreover, e is amongst its generators over $E\mathbb{Q}_*(F^*)$, so by Theorem 3.7 it consists of the entire Hopf ring. $\square$

We now explain how to relate $E_*^{\mathbb{Q}}(F_*)$ to $E_*(F_*)$, since our aim is to prove that the model is precise for a wide range of examples.

Because $E_*(F_*)$ contains both the ring ring $E_*[F^*]$ and the elements b_n, its image in $E\mathbb{Q}_*(F_*)$ under rationalisation must contain $E_*^{\mathbb{Q}}(F_*)$ by virtue of Definition 4.1. But the kernel of the rationalisation homomorphism is the torsion submodule T, from which we obtain a canonical inclusion

$$\kappa_F : E_*^{\mathbb{Q}}(F_*) \to E_*(F_*)/T. \tag{4.1}$$

We therefore seek to understand the circumstances under which $E_*(F_*)$ is torsion free and κ_F is epic; for then T is trivial, κ_F is an isomorphism, and our model is precise. In fact κ_F is always a *rational* isomorphism by the second part of Proposition 4.2.

At this point we can best proceed by considering several individual cases on their merits. We must keep in mind our blanket assumptions of Section 1 concerning the coefficient ring of F, and we leave it to the reader to check that each of our examples does indeed conform to those requirements. We begin with MU and BP.

Theorem 4.3. *If F is either MU or BP then κ_F induces an isomorphism of Hopf rings $E^{\mathbb{Q}}(F_*) \to E_*(F_*)$ for all E.*

Proof. As usual we begin with the case when E is the integral Eilenberg–MacLane spectrum H, for which we appeal to the beautiful work of Chan [2]; the result then follows for arbitrary E by the collapse of the Atiyah–Hirzebruch spectral sequence. Throughout our proof, F will denote either MU or BP.

Chan uses an inductive Rothenberg–Steenrod spectral sequence argument to deduce that $H_*(F_*)$ is torsion free. It therefore suffices to show that, for all primes p, the elements e and b_n generate $H_*(F_*; \mathbb{F}_p)$ over the ring ring $\mathbb{F}_p[F^*]$, since κ_F must then be epic. We proceed by a routine induction on homological dimension, extracted from Chan's calculations.

The hypothesis clearly holds for $H_0(F_*; \mathbb{F}_p) = \mathbb{F}_p[F^*]$, so we assume it demonstrated for $H_m(F_r; \mathbb{F}_p)$, for all r and all $m < k$. Chan shows that $H_*(F_r; \mathbb{F}_p)$ is a polynomial algebra for r even and an exterior algebra for r odd and considers the Rothenberg–Steenrod spectral sequences

$$E^2_{*,*} = \mathrm{Tor}^{H_*(F_r;\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_*(F'_{r+1}; \mathbb{F}_p), \tag{4.2}$$

which here always collapse. We consider the cases of r even and r odd separately.

If r is even and $H_*(F_r; \mathbb{F}_p)$ is polynomial on generators g_i, then (4.2) shows that the elements $e \circ g_i$ are exterior generators of $H_*(F_{r+1}; \mathbb{F}_p)$. Since $e\,\circ$ raises homological dimension by 1, the inductive step follows immediately. If r is odd and $H_*(F_r; \mathbb{F}_p)$ is exterior on generators g_i, then the E^∞-term of (4.2) is a divided power algebra generated by the $e \circ g_i$. We write $\gamma_n(x)$ for the nth divided power of x, and note that algebra generators are given by the γ_{p^j}, since we are working over $\mathbb{F}_p$. We therefore consider an element $\gamma_{p^j}(e \circ g_i)$ of dimension k. Then g_i has dimension less than k, whence the inductive hypothesis implies that $e \circ g_i$ may be written as a linear sum of elements $[y] \circ b^I$, with $y \in F^*$ and $b^I = b_{p^0}^{\circ i_0} \circ b_{p^1}^{\circ i_1} \circ \cdots$ for some finite sequence of nonnegative integers $I = (i_0, i_1, \ldots)$. Evaluation of the p^jth iterated coproduct shows that $\gamma_{p^j}([y] \circ b^I)$ represents the Hopf ring element $[y] \circ b^{\sigma_j(I)}$ modulo elements of lower bar filtration, where $\sigma_j(I) = (0, \ldots, 0, i_0, i_1, \ldots)$ (the sequence I shifted to the right by j zeros). This calculation clearly extends linearly, and so completes the induction. $\square$

Corollary 4.4. *Let F be any spectrum representing a Landweber exact cohomology theory. Then κ_F induces an isomorphism of Hopf rings $E^Q(F_*) \to E_*(F_*)$ for all E.*

Proof. We need to show that $E_*(F_*)$ is torsion free and κ_F is epic for any such F. A careful examination of the arguments of [5] shows that these follow from the Landweber exactness properties of F plus the fact that $H_*(MU_*)$ is torsion free, which we have from [2]. $\square$

These two proofs illustrate one of the prime assets of our model, and some further comment is necessary.

In [5], Hopkins and the first author deduce the alternative theorem that in the circumstances of Corollary 4.4 the Ravenel and Wilson model $E^R_*(F_*)$ is precise. But major effort is required to prove that the model is torsion free and maps monomorphically into the geometric Hopf ring $E_*(F_*)$; for this purpose, [5] insists on the results of [6] as prerequisites. In contrast, the route we have charted above avoids any reliance on $E^R_*(F_*)$ and [6] and is considerably simpler to boot, highlighting our remarks in Section 1 concerning the practical drawbacks of the Ravenel and Wilson model. We shall explore the connections between the models $E^Q_*(F_*)$ and $E^R_*(F_*)$ in some detail in Section 6.

Examples of spectra F to which Corollary 4.4 applies include the complex K-spectrum, the elliptic cohomology spectrum, *Ell*, and the Johnson–Wilson spectra $E(n)$, all of which have periodic coefficient groups.

5. Bell polynomials and examples

In this section we explain how to perform explicit computations in $E_*^Q(F_*)$ with the help of the conjugate Bell polynomials, and give examples from complex K-theory. We recall that w denotes the generator of $E_2(\mathbb{C}P^\infty)$, and passes to the element b_1 in each of the four Hopf rings of (3.1); moreover, it coincides with $e \circ e$ therein.

By virtue of Corollaries 3.4 and 3.8, we can express b_n in $E\mathbb{Q}_{2n}(F\mathbb{Q}_2)$ as a polynomial in w with respect to the operations $+$, $*$ and $\circ$, and henceforth we may write $b_n(w)$ to emphasise this fact. Explicit formulae are provided by the relations (2.6), and we shall now discuss them in detail.

As they stand, however, and because of the distinction between the underlying ring rings documented in (3.6), they hold good only in $E_*(F\mathbb{Q}_*)$ and $E\mathbb{Q}_*(F\mathbb{Q}_*)$. To convert them into a form applicable to $E\mathbb{Q}_*(F_*)$, we must appeal to (3.5) and deduce that

$$\left[\frac{f_{i-1}}{i!}\right] \circ e = \frac{1}{i!}[f_{i-1}] \circ e$$

in $E\mathbb{Q}_*(F\mathbb{Q}_*)$ for all $i > 0$ and f_{i-1} in F_{2i-2}. Then we may rewrite (2.6) as

$$b(s) = \prod_{i \geq 1} \left\{ \frac{1}{i!}[f_{i-1}] \circ (x_*(\beta(s)))^{\circ i} \right\}. \tag{5.1}$$

Of course, both (2.6) and (5.1) obtain in $E\mathbb{Q}_*(F\mathbb{Q}_*)$.

Elegant though such expressions may appear, their true virtue lies in the possibility of massaging them into a form more suited to practical computation. The following three lemmas are geared to this task.

We recall our discussion of the rationalisation $F\mathbb{Q}^*(-)$ in Section 2, and write

$$\mathrm{ex}(r) : \mathbb{C}P^\infty \to F\mathbb{Q}_2 \tag{5.2}$$

for the map representing $f_{r-1}x^r/r!$, the rth component of $\exp^F(x)$ in $F\mathbb{Q}^2(\mathbb{C}P^\infty)$. We then define the composition $c(m)$ by

$$\mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \to F\mathbb{Q}_2 \times \cdots \times F\mathbb{Q}_2 \to F\mathbb{Q}_2, \tag{5.3}$$

where the first map is the m-fold diagonal, the second map is $\mathrm{ex}(r)$ on the rth factor for all $1 \leq r \leq m$, and the third map is loop sum. In these terms, (2.3) may be restated thus.

Lemma 5.1. *For all $m \geq n$, the homomorphism induced by $c(m)$ acts such that*

$$c(m)_*(\beta_n) = b_n$$

in $E_{2n}(F\mathbb{Q}_2)$.

Henceforth, it simplifies notation to insist that m is chosen to be n in the above formula. Our aim therefore becomes to understand $c(n)_*$, for which purpose it will be most convenient to take advantage of Corollary 3.5 and work in $E\mathbb{Q}_*(-)$.

Since $E\mathbb{Q}_*(\mathbb{C}P^\infty)$ is isomorphic to the polynomial algebra $E\mathbb{Q}_*[w]$ (by applying the homology version of (2.1)), we may express each β_n as a polynomial in w; this is the normalised conjugate Bell polynomial $\beta_n(w)$ of [8]. In fact the $\beta_n(w)$ arise in various purely combinatorial situations, and may be described in terms of the coefficients of either the exponential series $\exp^E(X)$, or its compositional inverse $\log^E(X)$. The first three examples are given by

$$\beta_1(w) = w, \qquad \beta_2(w) = \frac{1}{2!}(w^2 - e_1 w),$$

$$\beta_3(w) = \frac{1}{3!}(w^3 - 3e_1 w^2 + (3e_1^2 - e_2)w),$$

and with ever-increasing effort (but without recourse to specialised combinatorial information) the list may be extended indefinitely.

Referring back to (5.3), and foreshadowing (1.1), the action of the diagonal is given by

$$\delta(\beta_n(w)) = \sum \binom{n}{p_1, \ldots, p_n} \beta_{p_1}(w) \otimes \cdots \otimes \beta_{p_n}(w). \tag{5.4}$$

Combinatorialists may choose to rewrite this formula in terms of the divided power property for $\beta_n(w_1 + \cdots + w_n)$, where w_i denotes $1 \otimes \cdots \otimes w \otimes \cdots \otimes 1$ with w in the ith position.

Lemma 5.2. *The induced homomorphism,*

$$\mathrm{ex}(r)_* : E\mathbb{Q}_*(\mathbb{C}P^\infty) \longrightarrow E\mathbb{Q}_*(F\mathbb{Q}_2)$$

acts on the element w^m *according to the rule*

$$\mathrm{ex}(r)_*(w^m) = \frac{1}{r!}[f_{r-1}] \circ x_*^r(w^m).$$

Proof. This follows at once from the definitions of the maps involved. □

We remind the reader that $x_*^r(w^m)$ and $(x_*(w^m))^{\circ r}$ are the same homology class.

Lemma 5.3. *The induced homomorphism*

$$x_*^r : E\mathbb{Q}_*(\mathbb{C}P^\infty) \longrightarrow E\mathbb{Q}_*(F\mathbb{Q}_2)$$

acts on the element w^m *according to the rule*

$$x_*^r(w^m) = \begin{cases} 0 & \text{if } m \not\equiv 0 \bmod r, \\ \dfrac{m!}{d!}(w^{\circ r})^d & \text{if } m = rd. \end{cases}$$

Proof. Recalling the comments following (2.5) and the genesis of the element w, it suffices to consider the map $x^r : \mathbb{C}P^\infty \to H(\mathbb{Q}, 2r)$. The stated formula then follows by repeated application of (3.3). □

The above result suggests the following notation. Write the conjugate Bell polynomial $\beta_n(w)$ as $\sum_{m=1}^{n} \beta_{n,m} w^m$, and define the modified polynomial $b_{n/r}(w)$ by

$$b_{n/r}(w) = \sum_{d=1}^{\lfloor n/r \rfloor} \frac{(rd)!}{d!} \beta_{n,rd} w^d \tag{5.5}$$

for each $1 \le r \le n$; thus $b_{n/1}(w)$ is $\beta_n(w)$. In addition, let $b_{n/r}(w)$ be zero whenever $r \ge n$, and $[0]$ (which acts as 1 in our multiplicative notation) whenever n is zero. Note that each $\beta_{n,m}$ lies in the rational coefficient group $E\mathbb{Q}_{2(n-m)}$. Employing (5.5), we are able to collect together the three preceding lemmas and immediately obtain our main result.

Theorem 5.4. *The polynomial $b_n(w)$ is given by*

$$b_n(w) = \sum \binom{n}{p_1, \ldots, p_n} \left\{ \prod_{r=1}^{n} \frac{1}{r!} [f_{r-1}] \circ b_{p_r/r}(w^{or}) \right\}$$

in $E\mathbb{Q}_{2n}(F\mathbb{Q}_2)$.

The first three such polynomials are easily computed to be

$$b_1(w) = w, \qquad b_2(w) = \frac{1}{2!}(w^2 - e_1 w + [f_1] \circ w^{o2}),$$

$$b_3(w) = \frac{1}{3!}(w^3 - 3e_1 w^2 + (3e_1^2 - e_2) w + 3w([f_1] \circ w^{o2})$$
$$- 3e_1 [f_1] \circ w^{o2} + [f_2] \circ w^{o3}).$$

If both E and F are the MU spectrum, these polynomials are universal, and give rise to interesting generalisations of the associated sequences of umbral calculus [9]; but we shall pursue this idea elsewhere.

The following corollary to Theorem 5.4 will be useful.

Corollary 5.5. *In the case when E is H, the polynomial $b_n(w)$ is given by*

$$b_n(w) = \sum \binom{n}{q_1, 2q_2, \ldots, nq_n} \prod_{r=1}^{n} \frac{1}{r!\, q_r!} ([f_{r-1}] \circ w^{or})^{q_r}$$

in $H\mathbb{Q}_{2n}(F\mathbb{Q}_2)$.

Proof. In this case, $\beta_n(w)$ reduces to w^n for all n. □

As a simple illustration of these methods, we conclude by outlining some calculations in the case when F is the complex K spectrum. Although not difficult, this example seems only to have been considered (and from an alternative viewpoint) in the currently unpublished thesis [10].

Bott periodicity tells us that the spaces $\boldsymbol{K}_r$ are given by $\mathbb{Z} \times BU$ when r is even and U when r is odd, so that the Hopf algebras $E_*(\boldsymbol{K}_r)$ are well known. It is not our aim here to rederive these facts from scratch; rather, we wish to place known results in a Hopf ring setting. However, if we allow ourselves to assume that only the coefficient ring $K_* = \mathbb{Z}[u, u^{-1}]$ and the exponential series

$$\exp^K(x) = \sum_{i \geq 1} \frac{1}{i!} u^{i-1} x^i = u^{-1}(e^{ux} - 1),$$

are given, then by virtue of Corollary 4.4 we do in fact recover a calculation of $E_*(\boldsymbol{K}_r)$ for each of the spaces $\boldsymbol{K}_r$.

We begin with the usual special case, and apply Proposition 3.6 to describe $H\mathbb{Q}_*(\boldsymbol{K}_*)$.

Proposition 5.6. *As $\mathbb{Q}$-algebras with respect to the $*$ product, we have that*

$$H\mathbb{Q}_*(\boldsymbol{K}_{2r}) \cong \mathbb{Q}[[\pm u^{-r}], [u^{n-r}] \circ w^{\circ n}\colon n = 1, 2, \ldots]$$

and

$$H\mathbb{Q}_*(\boldsymbol{K}_{2r+1}) \cong \wedge_{\mathbb{Q}}([u^{n-r}] \circ w^{\circ n} \circ e\colon n = 0, 1, \ldots)$$

for each integer r. The $\circ$ product is determined by the formula

$$\left(\prod_i ([u^{m_i - r}] \circ w^{\circ m_i})^{g_i}\right) \circ \left(\prod_j ([u^{n_j - s}] \circ w^{\circ n_j})^{h_j}\right)$$
$$= \sum_A \left(\prod_{i,j} \frac{g_i!\, h_j!}{a_{i,j}!} ([u^{m_i + n_j - (r+s)}] \circ w^{\circ (m_i + n_j)})^{a_{i,j}}\right) \tag{5.6}$$

on $*$ monomials, where the sum is taken over all matrices $A = (a_{i,j})$ of positive integers satisfying $\sum_j a_{i,j} = g_i$ and $\sum_i a_{i,j} = h_j$, and is to be interpreted as zero if no such A exist.

Proof. The formula (5.6) follows by repeated application of (3.3) and the Hopf ring distributivity axiom, and suffices to establish the proposed structure. □

We remark that the $*$ monomials which occur in both sides of (5.6) have degree $s = \sum_i g_i = \sum_j h_j$, and that the generalisations to $E\mathbb{Q}_*(\boldsymbol{K}_*)$ are obtained by applying $E_* \otimes$ throughout.

To proceed, we make two observations. Firstly, the exponential series may be used to rephrase the formal group law for K as

$$\exp^K(x + y) = \exp^K(x) + \exp^K(y) + u \exp^K(x) \exp^K(y) \tag{5.7}$$

in $K\mathbb{Q}^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Secondly, by substituting u^{r-1} for f_{r-1} in Corollary 5.5 and writing I for the augmentation ideal as in [6], we obtain

$$b_n(w) \equiv \frac{1}{n!}[u^{n-1}] \circ w^{\circ n} \tag{5.8}$$

in $H\mathbb{Q}_{2n}(\boldsymbol{K}_2)$, modulo the $*$-decomposables $(IH\mathbb{Q}_*(\boldsymbol{K}_*))^2$.

Theorem 5.7. *As algebras with respect to the $*$ product, we have that*

$$H_*(\boldsymbol{K}_{2r}) \cong \mathbb{Z}[[\pm u^{-r}], [u^{1-r}] \circ b_n(w) \colon n = 1, 2, \ldots]$$

and

$$H_*(\boldsymbol{K}_{2r+1}) \cong \wedge([u^{1-r}] \circ b_n(w) \circ e \colon n = 0, 1, \ldots)$$

for each integer r. The $\circ$ product obeys

$$\left(\prod_i^I ([u^{1-r}] \circ b_{m_i}(w))^{g_i}\right) \circ \left(\prod_{j=1}^J ([u^{1-s}] \circ b_{n_j}(w))^{h_j}\right)$$

$$\equiv \sum_A \left(\prod_{i,j} \frac{g_i!\,h_j!}{a_{i,j}!} \binom{m_i + n_j}{m_i} ([u^{1-(r+s)}] \circ b_{m_i+n_j}(w))^{a_{i,j}}\right) \tag{5.9}$$

modulo $(IH_*(\boldsymbol{K}_*))^{s+1}$.

Proof. The algebras with the given generators are polynomial and exterior respectively, by virtue of Proposition 5.6 and (5.8); and they are certainly subalgebras of our model $H_*^{\mathbb{Q}}(\boldsymbol{K}_*)$, which we know to be precise from Corollary 4.4 It therefore suffices to demonstrate that they are closed with respect to $\circ$ product. So far as the generators are concerned, we may inductively apply

$$\binom{m+n}{m} b_{m+n}(w) = [u] \circ b_m(w) \circ b_n(w) + b_m(w) b_n(w)$$

$$+ \sum_{0 < i+j < m+n} b_{m-i}(w) b_{n-j}(w) ([u] \circ b_i(w) \circ b_j(w)),$$

a formula which holds for all integers $m, n \geq 0$ and is established by appealing to (5.7) (it may also be verified directly from Theorem 5.4). Thus

$$b_m(w) \circ b_n(w) \equiv \binom{m+n}{n} [u^{-1}] \circ b_{m+n}(w)$$

modulo $(IH_*(\boldsymbol{K}_*))^2$. Closure for general monomials then follows by applying the distributivity axiom.

Now consider the two sides of (5.9). If we apply (5.8) we obtain the two sides of (5.6) modulo $(IH\mathbb{Q}_*(\boldsymbol{K}_*))^{s+1}$. The required result is thus a consequence of the commutativity of the square

$$\begin{tikzcd}
H_*(\boldsymbol{K}_*) \arrow[r] \arrow[d] & H\mathbb{Q}_*(\boldsymbol{K}_*) \arrow[d] \\
H_*(\boldsymbol{K}_*)/(IH_*(\boldsymbol{K}_*))^{s+1} \arrow[r] & H\mathbb{Q}_*(\boldsymbol{K}_*)/(IH\mathbb{Q}_*(\boldsymbol{K}_*))^{s+1}
\end{tikzcd}$$

and the fact that the lower map is a monomorphism. $\square$

The generalisation of Theorem 5.7 to $E_*(\boldsymbol{K}_*)$ follows, as usual, by applying $E_* \otimes$ throughout and replacing the homology generators $b_n(w)$ by their E-theory counterparts.

To conclude our analysis of the K spectrum, we observe that $\circ\, u^{-1}$ defines a map $\boldsymbol{K}_{2r} \to \boldsymbol{K}_{2r+2}$ which has $\circ\, u$ as inverse. In homology this induces the homomorphism $\circ\,[u^{-1}]$, which is visibly an isomorphism of $*$ Hopf algebras by virtue of (5.8). If we write $\boldsymbol{K}'_{2r}$ as BU_r and $\boldsymbol{K}_{2r+1}$ (which is connected) as U_r, we arrive at the following corollary to the general form of (5.8).

Corollary 5.8. *As algebras with respect to the $*$ product, we have that*

$$E_*(BU_r) \cong E_*[[u^{1-r}] \circ b_n(w)\colon n = 1, 2, \dots]$$

and

$$E_*(U_r) \cong \wedge_{E_*}([u^{1-r}] \circ b_n(w) \circ e\colon n = 0, 1, \dots)$$

for each integer r.

We have not yet established that the Ravenel and Wilson model $E^{\mathrm{R}}_*(\boldsymbol{K}_*)$ is precise; this will follow from our deliberations in Section 6, although it may be deduced directly from the definitions, and is stated explicitly in [5, 10].

6. Relations with the Ravenel and Wilson model

We conclude by investigating the relationship between $E^{Q}_*(\boldsymbol{F}_*)$ and the Ravenel and Wilson model $E^{R}_*(\boldsymbol{F}_*)$. We outlined the construction of this model in Section 1, and described the Hopf ring homomorphism $\tau_F\colon E^{R}_*(\boldsymbol{F}_*) \to E_*(\boldsymbol{F}_*)$; much attention has been devoted in the literature to showing that τ_F is an isomorphism in many standard cases.

Our first result gives us a natural map of $E^{R}_*(\boldsymbol{F}_*)$ onto $E^{Q}_*(\boldsymbol{F}_*)$. We write π_F for the projection $E_*(\boldsymbol{F}_*) \to E_*(\boldsymbol{F}_*)/T$ onto the torsion free quotient Hopf ring, and recall the monomorphism $\kappa_F\colon E^{Q}_*(\boldsymbol{F}_*) \to E_*(\boldsymbol{F}_*)/T$ introduced in Section 4.

Proposition 6.1. *There is a commutative diagram*

$$\begin{array}{ccc} E^R_*(F_*) & \xrightarrow{\tau_F} & E_*(F_*) \\ {\scriptstyle \sigma_F}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ E^Q_*(F_*) & \xrightarrow{\kappa_F} & E_*(F_*)/T \end{array} \tag{6.1}$$

in which σ_F is an epimorphism of Hopf rings.

Proof. By definition, $E^R_*(F_*)$ is the free $E_*[F^*]$-Hopf ring on e and the elements b_n, subject to certain relations arising from the interaction of the formal group laws for E and F. These relations, or rather their rationalisations, must also hold in $E\mathbb{Q}_*(F_*)$. There is thus a natural epimorphism σ_F which makes the diagram commute as claimed. □

The following corollary generalises the proofs of Theorem 4.3 and Corollary 4.4.

Corollary 6.2. *If the composite*

$$E^R_*(F_*) \xrightarrow{\tau_F} E_*(F_*) \xrightarrow{\pi_F} E_*(F_*)/T$$

is epic, then κ_F is an isomorphism. Hence, if τ_F is epic and $E_*(F_*)$ is torsion free, then κ_F yields an isomorphism $E^Q_*(F_*) \cong E_*(F_*)$.

Proof. This follows from the diagram (6.1). □

It is clearly important to understand when the kernel of σ_F is trivial, for then σ_F is an isomorphism. To examine this question, we must consider the Ravenel and Wilson model for rational E.

Proposition 6.3. *The homomorphism*

$$\tau_F : E\mathbb{Q}^R_*(F_*) \longrightarrow E\mathbb{Q}_*(F_*)$$

is an isomorphism of Hopf rings.

Proof. It suffices to prove this for $E = H$ since we have the identities

$$E\mathbb{Q}_*(F_*) = E\mathbb{Q}_* \otimes_{\mathbb{Q}} H\mathbb{Q}_*(F_*) \quad \text{and} \quad E\mathbb{Q}^R_*(F_*) = E\mathbb{Q}_* \otimes_{\mathbb{Q}} H\mathbb{Q}^R_*(F_*).$$

The result is clearly true in dimension zero; it is in higher dimensions that we have something to prove. Theorem 3.7 shows that τ_F is epic, so it suffices to show that τ_F is monic.

To study $H\mathbb{Q}_*^R(F_*)$ we must review the formal group law relations of [6, 3.9(i)]. Writing the formal group law for F as

$$F(X,Y) = \sum_{i,j\geq 0} a_{i,j}^F X^i Y^j, \quad a_{i,j}^F \in F_{2(i+j-1)},$$

then the Hopf ring relations with which we are concerned arise by equating coefficients in the power series identity

$$\sum_{n\geq 0} b_n(s+t)^n = \prod_{i,j\geq 0} [a_{i,j}^F] \circ \Big(\sum_{k\geq 0} b_k s^k\Big)^{\circ i} \circ \Big(\sum_{l\geq 0} b_l t^l\Big)^{\circ j}. \tag{6.2}$$

Considering right-hand side monomials of the form $s^p t^q$, where $p+q=n$ and $p,q>0$, we deduce that each b_n may be expressed as a $\mathbb{Q}$-linear combination of $*$ and $\circ$ products of elements $[a_{i,j}^F]$ and b_k, where $1 \leq k < n$. Since $e \circ e = b_1$, it follows that $H\mathbb{Q}_*^R(F_*)$ is generated as a Hopf ring $\mathbb{Q}[F^*]$ by the single element e. The result is now a consequence of Theorem 3.7. $\square$

Parallel to this result, although it will not be needed here, is the fact that the Hopf ring map

$$\tau_{FQ}: E_*^R(F\mathbb{Q}_*) \to E_*(F\mathbb{Q}_*)$$

is also an isomorphism. We may check this by direct calculation in the case when F is H, and then proceed to general F by using the equivalence (2.4) and the naturality of the relations (6.2) with respect to maps of complex oriented ring spectra.

Corollary 6.4. *The epimorphism $\sigma_F: E_*^R(F_*) \to E_*^Q(F_*)$ is rational isomorphism, so that the kernel of σ_F consists of the torsion elements in $E_*^R(F_*)$.*

Proof. The homomorphism $\tau_F: E_*^R(F_*) \to E_*(F_*)$ is a rational isomorphism by Proposition 6.3. So also are π_F and κ_F by Proposition 4.2; the result therefore follows from the diagram (6.1). $\square$

Applying (6.1) again, we deduce two further corollaries.

Corollary 6.5. *If $E_*^R(F_*)$ is torsion free then τ_F is a monomorphism.*

Corollary 6.6. *Suppose that $E_*^R(F_*)$ is torsion free and τ_F is an epimorphism; then τ_F, π_F, σ_F and κ_F are all isomorphisms.*

The above results fulfill our aim of describing the relationship between the models $E_*^Q(F_*)$ and $E_*^R(F_*)$, and illustrate the fundamental distinction provided by the possibility of torsion in the latter. In concert with the calculational advantages demonstrated in Section 5, they add further support to our claims for the potential superiority of $E_*^Q(F_*)$, at least in cases where $E_*(F_*)$ is torsion free.

In fact one of the most serious drawbacks of $E_*^R(F_*)$ is the difficulty of determining whether it does, or does not, have torsion. Returning to the comments at the end of Section 4, we suggest that the arguments of [5] which show that τ_F is isomorphic for a Landweber exact F actually tell us more about $E_*^R(F_*)$ than about $E_*(F_*)$.

References

[1] J.F. Adams, Stable homotopy and generalised homology, Univ. of Chicago Press, Chicago, Illinois and London, 1974.

[2] K. Chan, A simple proof that the unstable (co) homology of the Brown–Peterson spectrum is torsion free, J. Pure Appl. Algebra 26 (1982) 155–157.

[3] F.W. Clarke, J.R. Hunton and N. Ray, Extensions of umbral calculus II; double delta operators and Hattori–Stong theorems, in preparation.

[4] A. Dold, On general cohomology, Lectures at the Nordic Summer School, chs. 1–9, Matematisk Institut, Aarhus Universitet, 1968.

[5] M.J. Hopkins and J.R. Hunton, On the structure of spaces representing a Landweber exact cohomology theory, Topology, 34 (1995) 29–36.

[6] D.C. Ravenel and W.S. Wilson, The Hopf ring for complex cobordism, J. Pure Appl. Algebra 9 (1977) 241–280.

[7] N. Ray, Some results in generalized homology, K-theory and bordism, Proc. Cambridge Phil. Soc. 71 (1972) 283–300.

[8] N. Ray, Symbolic calculus: a 19th century approach to MU and BP, in: Homotopy Theory, LMS Lecture Note Series, Vol. 117, J.D.S. Jones and E. Rees, eds., (1987) 195–238.

[9] S.M. Roman and G.-C. Rota, The umbral calculus, Adv. Math. 27 (1978) 95–188.

[10] N. Strickland, Ph.D. Thesis, Manchester University, 1992.

[11] D. Sullivan, Genetics of homotopy theory and the Adams conjecture, Ann. Math. 100 (1974) 1–79.